Burgers' flows as Markovian diffusion processes

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We analyze the unforced and deterministically forced Burgers equation in the framework of the (diffusive) interpolating dynamics that solves the so-called Schrödinger boundary data problem for random matter transport. This entails an exploration of the consistency conditions that allow one to interpret dispersion of passive contaminants in Burgers flow as a Markovian diffusion process. In general, the usage of a continuity equation $\partial_t\rho=-\vec{\nabla}(\vec{v}\rho)$, where $\vec{v}=\vec{v}(\vec{x},t)$ stands for the Burgers field and ρ is the density of transported matter, is at variance with the explicit diffusion scenario. Under these circumstances, we give a complete characterization of the diffusive transport that is governed by Burgers velocity fields. The result extends both to the approximate description of the transport driven by an incompressible fluid and to motions in an infinitely compressible medium. Also, in conjunction with the Born statistical postulate in quantum theory, it pertains to the probabilistic (diffusive) counterpart of the Schrödinger picture quantum dynamics. We give a generalization of this dynamical problem to cases governed by nonconservative force fields when it appears indispensable to relax the gradient velocity field assumption. The Hopf-Cole procedure has been appropriately generalized to yield solutions in that case. [S1063-651X(97)04302-X]

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I. BURGERS VELOCITY FIELDS AND THE RELATED STOCHASTIC TRANSPORT PROCESSES

The Burgers equation [1,2] recently has acquired considerable popularity in a variety of physical contexts [3–20]. An exhaustive discussion of its role in acoustic turbulence and gravitational contexts, where the emergence of shock pressure fronts is crucial, can be found in Ref. [17].

As is well known, the logarithmic Hopf-Cole transformation [2] allows one to replace the nonlinear problem (nonlinear diffusion equation [1]) by a linear parabolic equation. Because of this equivalence all gradient-type solutions of the Burgers equation are known exactly.

At the moment we shall preserve the gradient form restriction for Burgers velocity fields, but consider a more general form of the Burgers equation that accounts for an external force field $\vec{F}(\vec{x},t)$:

$$\partial_t\vec{v}+(\vec{v}\cdot\vec{\nabla})\vec{v}=\nu\Delta\vec{v}+\vec{F}(\vec{x},t). \tag{1}$$

Let us mention that many recent investigations were devoted to the analysis of curl$\vec{v}=\vec{0}$ solutions that are statistically relevant in view of the random initial data choice and/or inclusion of the random forcing term (the random potential in the related Parisi-Kardar equation [11]).

However, irrespective of whether we do or do not need the statistical input, an issue of matter transport driven by those nonlinear velocity fields requires the knowledge of an exact evolution of concentration and/or density fields, much in the spirit of early hydrodynamical studies of advection and diffusion of passive tracers [21,22]; see also [23]. This particular issue is addressed in the present paper, under a simplifying assumption of nonrandom initial data and deterministic force fields.

Following the traditional motivation (applicable both to incompressible and infinitely compressible liquids [1]), we regard the stochastic diffusion process as a primary phenomenon responsible for the emergence of Eq. (1) and thus justifying the ''nonlinear diffusion equation'' phrase in this context.

Knowing the Burgers velocity fields, one is tempted to ask what is the particular dynamics (of matter or probability density fields) that is consistent with the chosen Burgers velocity field evolution. The corresponding passive scalar (tracer or contaminant) advection-in-a-flow problem [14,11,16] is normally introduced through the parabolic dynamics:

$$\partial_tT+(\vec{v}\cdot\vec{\nabla})T=\nu\Delta T; \tag{2}$$

see, e.g., [21–23]. For incompressible fluids, Eq. (2) coincides with the conventional Fokker-Planck equation for the diffusion process. This feature does not persist in the compressible case.

While looking for the stochastic implementation of the microscopic (molecular) dynamics, Eq. (2) [11,16,23,24], it is assumed that the ''diffusing scalar'' (contaminant in the lore of early statistical turbulence models) obeys an Itô equation:

$$d\vec{X}(t)=\vec{v}(\vec{x},t)dt+\sqrt{2\nu}d\vec{W}(t), \tag{3}$$

$$\vec{X}(0)=\vec{x}_0\rightarrow\vec{X}(t)=\vec{x},$$

where the given forced Burgers velocity field is perturbed by the noise term representing a molecular diffusion. In the (by now conventional) Itô representation of diffusion-type random variable $\vec{X}(t)$ one explicitly refers to the standard Brownian motion (e.g., the Wiener process) $\sqrt{2\nu}\vec{W}(t)$, instead of the usually adopted formal white noise integral $\int_0^t\vec{\eta}(s)ds$, coming from the Langevin-type version of Eqs. (3).

Under these premises, while taking for granted that *there is* a diffusion process involved, we cannot view Eqs. (1)–(3) as completely independent (disjoint) problems: the velocity field $\vec{v}$ cannot be quite arbitrarily inferred from Eq. (1) or any other velocity-defining equation without verifying the *consistency* conditions, which would allow one to associate Eqs. (2) and (3) with a well defined random dynamics, and Markovian diffusion in particular [25,26].

In connection with the usage of Burgers velocity fields (with or without external forcing), which in Eqs. (3) clearly are intended to replace the standard *forward drift* of the would-be involved Markov diffusion process, we have not found in the literature any attempt to resolve apparent contradictions arising if Eqs. (2) and/or (3) are defined by means of Eq. (1). In particular, the usage of a continuity equation $\partial_t \rho = -\vec{\nabla}(\vec{v}\rho)$, where $\vec{v}=\vec{v}(\vec{x},t)$ stands for the Burgers field and ρ is the density of transported matter, is at variance with the explicit diffusion scenario. Also, an issue of the necessary *correlation* (cf. [16], Chap. 7.3, devoted to the turbulent transport and the related dispersion of contaminants) between the probabilistic Fokker-Planck dynamics of the diffusing tracer, and this of the passive tracer (contaminant) concentration [Eq. (2)], has been left aside in the literature.

Moreover, rather obvious hesitation could have been observed in attempts to establish the most appropriate matter transport rule, if Eqs. (1)–(3) are adopted. Depending on the particular phenomenological departure point, one either adopts the standard continuity equation [3,4], that is certainly valid to a high degree of accuracy in the low viscosity limit (we refer to the standard terminology that comes from viscous fluid models; here, ν stands for the diffusion constant) $\nu \downarrow 0$ of Eqs. (1)–(3), but incorrect on mathematical grounds *if* there is a diffusion involved *and* simultaneously a solution of Eq. (1) is interpreted as the respective *current* velocity of the flow: $\partial_t \rho(\vec{x},t) = -\vec{\nabla}\cdot[\vec{v}(\vec{x},t)\rho(\vec{x},t)]$. Alternatively, following the white noise calculus tradition telling that the stochastic integral $\vec{X}(t)=\int_0^t \vec{v}(\vec{X}(s),s)ds + \int_0^t \vec{\eta}(s)ds$ implies the Fokker-Planck equation, one adopts [24]: $\partial_t \rho(\vec{x},t) = \nu\Delta\rho(\vec{x},t) - \vec{\nabla}\cdot[\vec{v}(\vec{x},t)\rho(\vec{x},t)]$, which is clearly problematic in view of the classic McKean's discussion of the propagation of chaos for the Burgers equation [27–29] and the derivation of the stochastic ''Burgers process'' in this context: ''the fun begins in trying to describe this Burgers motion as the path of a tagged molecule in an infinite bath of like molecules'' [27].

To put things on solid ground, let us consider a Markovian diffusion process, which is characterized by the transition probability density (generally inhomogeneous in space and time law of random displacements) $p(\vec{y},s,\vec{x},t), 0 \le s < t \le T$, and the probability density $\rho(\vec{x},t)$ of its random variable $\vec{X}(t), 0 \le t \le T$. The process is completely determined by these data. For clarity of discussion, we do not impose any spatial boundary restrictions, nor fix any concrete limiting value of T which, in principle, can be moved to infinity.

The conditions valid for any $\epsilon > 0$: (a) there holds

$$\lim_{t\downarrow s} \frac{1}{t-s}\int_{|\vec{y}-\vec{x}|>\epsilon} p(\vec{y},s,\vec{x},t)d^3x = 0,$$

(b) there exists a (forward) drift

$$\vec{b}(\vec{x},s) = \lim_{t\downarrow s} \frac{1}{t-s}\int_{|\vec{y}-\vec{x}|\le\epsilon} (\vec{y}-\vec{x})p(\vec{x},s,\vec{y},t)d^3y,$$

and (c) there exists a diffusion function (in our case it is simply a diffusion coefficient ν)

$$a(\vec{x},s) = \lim_{t\downarrow s} \frac{1}{t-s}\int_{|\vec{y}-\vec{x}|\le\epsilon} (\vec{y}-\vec{x})^2 p(\vec{x},s,\vec{y},t)d^3y,$$

are conventionally interpreted to define a diffusion process [25,26]. Under suitable restrictions the function

$$g(\vec{x},s) = \int p(\vec{x},s,\vec{y},T)g(\vec{y},T)d^3y \tag{4}$$

satisfies the backward diffusion equation [notice that the minus sign appears, in comparison with Eq. (2)]

$$-\partial_s g(\vec{x},s) = \nu\Delta g(\vec{x},s) + [\vec{b}(\vec{x},s)\cdot\vec{\nabla}]g(\vec{x},s). \tag{5}$$

Let us point out that the validity of Eq. (5) is known to be a *necessary* condition for the existence of a Markov diffusion process, whose probability density $\rho(\vec{x},t)$ is to obey the Fokker-Planck equation. Here, the new velocity field, named the forward drift of the process $\vec{b}(\vec{x},t)$, replaces the previously utilized Burgers field $\vec{v}(\vec{x},t)$:

$$\partial_t \rho(\vec{x},t) = \nu\Delta\rho(\vec{x},t) - \vec{\nabla}\cdot[\vec{b}(\vec{x},t)\rho(\vec{x},t)]. \tag{6}$$

The case of particular interest in the nonequilibrium statistical physics literature appears when $p(\vec{y},s,\vec{x},t)$ is a *fundamental solution* of Eq. (5) with respect to variables $\vec{y},s$ [25,26,30]; see, however, [31] for an alternative situation. Then, the transition probability density *also* satisfies the Fokker-Planck equation in the remaining $\vec{x},t$ pair of variables. Let us emphasize that these two equations form an adjoint pair, referring to the slightly counterintuitive for physicists, although transparent for mathematicians [33–37], issue of time reversal of diffusion processes.

After adjusting Eqs. (3) to the present context, $\vec{X}(t)=\int_0^t \vec{b}(\vec{X}(s),s)ds + \sqrt{2\nu}\vec{W}(t)$, we realize [35–38] that for any smooth function $f(\vec{x},t)$ of the random variable $\vec{X}(t)$ the conditional expectation value

$$\lim_{\Delta t\downarrow 0}\frac{1}{\Delta t}\left[\int p(\vec{x},t,\vec{y},t+\Delta t)f(\vec{y},t+\Delta t)d^3y - f(\vec{x},t)\right]$$

$$= (D_+ f)(\vec{X}(t),t) = [\partial_t + (\vec{b}\cdot\vec{\nabla}) + \nu\Delta]f(\vec{x},t), \tag{7}$$

$$\vec{X}(t) = \vec{x},$$

determines the forward drift $\vec{b}(\vec{x},t)$ (if we set components of $\vec{X}$ instead of f) and allows one to introduce the local field of (forward) accelerations associated with the diffusion process, which we constrain by demanding (see, e.g., Refs. [35–38] for prototypes of such dynamical constraints):

$$(D_+^2\vec{X})(t)=(D_+\vec{b})(\vec{X}(t),t)$$
$$=[\partial_t\vec{b}+(\vec{b}\cdot\vec{\nabla})\vec{b}+\nu\Delta\vec{b}](\vec{X}(t),t)$$
$$=\vec{F}(\vec{X}(t),t), \tag{8}$$

where, at the moment arbitrary, function $\vec{F}(\vec{x},t)$ may be interpreted as the external deterministic forcing applied to the diffusing system [32]. In particular, if we assume that drifts remain gradient fields, curl$\vec{b}=\vec{0}$, under the forcing, then those that are allowed by the prescribed choice of $\vec{F}(\vec{x},t)$ *must* fulfill the compatibility condition (notice the conspicuous absence of the standard Newtonian minus sign in this analog of Newton's second law)

$$\vec{F}(\vec{x},t)=\vec{\nabla}\Omega(\vec{x},t), \tag{9}$$

$$\Omega(\vec{x},t)=2\nu\left[\partial_t\Phi+\frac{1}{2}\left(\frac{\vec{b}^2}{2\nu}+\vec{\nabla}\cdot\vec{b}\right)\right].$$

This establishes the connection of the forward drift $\vec{b}(\vec{x},t)=2\nu\nabla\Phi(\vec{x},t)$ with the (Feynman-Kac; cf. [31,32]) potential $\Omega(\vec{x},t)$ of the chosen external force field. The latter connection, without invoking the Feynman-Kac formula, is frequently exploited in the theory of Smoluchowski-type diffusion processes, when the Fokker-Planck equation is transformed into the associated generalized diffusion equation.

One of distinctive features of Markovian diffusion processes with the positive density $\rho(\vec{x},t)$ is that the notion of the *backward* transition probability density $p_*(\vec{y},s,\vec{x},t)$ can be consistently introduced on each finite time interval $0\leqslant s<t\leqslant T$:

$$\rho(\vec{x},t)p_*(\vec{y},s,\vec{x},t)=p(\vec{y},s,\vec{x},t)\rho(\vec{y},s), \tag{10}$$

so that $\int\rho(\vec{y},s)p(\vec{y},s,\vec{x},t)d^3y=\rho(\vec{x},t)$ and $\rho(\vec{y},s)=\int p_*(\vec{y},s,\vec{x},t)\rho(\vec{x},t)d^3x$. This allows one to define (cf. [32,38–40] for a discussion of these concepts in the case of the most traditional Brownian motion and Smoluchowski-type diffusion processes)

$$\lim_{\Delta t\downarrow 0}\frac{1}{\Delta t}\left[\vec{x}-\int p_*(\vec{y},t-\Delta t,\vec{x},t)\vec{y}d^3y\right]$$
$$=(D_-\vec{X})(t)=\vec{b}_*(\vec{X}(t),t), \tag{11}$$

$$(D_-f)(\vec{X}(t),t)=[\partial_t+(\vec{b}_*\cdot\vec{\nabla})-\nu\Delta]f(\vec{X}(t),t).$$

Accordingly, the backward version of the dynamical constraint imposed on the local acceleration field reads

$$(D_-^2\vec{X})(t)=(D_+^2\vec{X})(t)=\vec{F}(\vec{X}(t),t), \tag{12}$$

where under the gradient-drift field assumption, curl$\vec{b}_*=0$, we have explicitly involved the forced Burgers equation [cf. Eq. (1)]:

$$\partial_t\vec{b}_*+(\vec{b}_*\cdot\vec{\nabla})\vec{b}_*-\nu\Delta\vec{b}_*=\vec{F}. \tag{13}$$

Here [32,35,36], in view of $\vec{b}_*=\vec{b}-2\nu\vec{\nabla}\ln\rho$, we deal with $\vec{F}(\vec{x},t)$ previously introduced in Eqs. (9). A notable consequence is that the Fokker-Planck equation (6) can be transformed to an *equivalent* form of

$$\partial_t\rho(\vec{x},t)=-\nu\Delta\rho(\vec{x},t)-\nabla[\vec{b}_*(\vec{x},t)\rho(\vec{x},t)], \tag{14}$$

which, however, describes a density evolution in the reverse sense of time.

At this point let us recall that Eqs. (5) and (6) form a natural adjoint pair of equations that determine the Markovian diffusion process in the chosen time interval $[0,T]$. Clearly, an adjoint of Eq. (14) reads:

$$\partial_s f(\vec{x},s)=\nu\Delta f(\vec{x},s)-[\vec{b}_*(\vec{x},s)\cdot\vec{\nabla}]f(\vec{x},s), \tag{15}$$

where

$$f(\vec{x},s)=\int p_*(\vec{y},0,\vec{x},s)f(\vec{y},0)d^3y, \tag{16}$$

to be compared with Eqs. (4), (5), and the previously mentioned passive scalar dynamics [Eq. (2)]; see also, e.g., [24]. Here, manifestly, the time evolution of the backward drift is governed by the Burgers equation, and the diffusion equation (15) is correlated [via the definition (10)] with the probability density evolution rule (14).

This pair *only* can be consistently utilized if the diffusion process is to be driven by forced (or unforced) Burgers velocity fields. Certainly, the continuity equation postulated to involve the Burgers field as the current velocity does not hold true in this context.

Let us point out that the study of diffusion in the Burgers flow may begin from first solving the Burgers equation (12) for a chosen external force field, next specifying the probability density evolution (14), and eventually ending with the corresponding "passive contaminant" concentration dynamics (15) and (16). All that is in perfect agreement with the heuristic discussion of the concentration dynamics given in Ref. [16], Chap. 7.3, where the "backward dispersion" problem with "time running backwards" was found necessary to *predict* the concentration.

All that means that Eqs. (1)–(3) can be reconciled in the framework set by Eqs. (4)–(16). Then, the "nonlinear diffusion equation" does indeed refer to consistent stochastic diffusion processes.

We are now at the point where the Burgers equation and the related matter transport can be consistently embedded in the general probabilistic framework of the so-called Schrödinger's boundary data (stochastic interpolation) problem [31,32,36,37,40–41], see also [42,43]. In this setting, the familiar Hopf-Cole transformation [2,44] of the Burgers equation into the generalized diffusion equation (yielding explicit solutions in the unforced case) receives a useful generalization.

Indeed, in that framework [31,32], the problem of deducing a suitable Markovian diffusion process was reduced to investigating the adjoint pairs of parabolic partial differential equations, like, e.g., Eqs. (5) and (6) or Eqs. (14) and (15). In the case of gradient drift fields this amounts to checking [this imposes limitations on the admissible force field potential, cf. also formula (9)] whether the Feynman-Kac kernel

$$k(\vec{y},s,\vec{x},t)=\int \exp\left[-\int_s^t c(\omega(\tau),\tau)d\tau\right]d\mu_{(\vec{x},t)}^{(\vec{y},s)}(\omega) \qquad (17)$$

is positive and continuous in the open space-time area of interest, and whether it gives rise to positive solutions of the adjoint pair of generalized heat equations:

$$\partial_t u(\vec{x},t)=\nu\Delta u(\vec{x},t)-c(\vec{x},t)u(\vec{x},t), \qquad (18)$$

$$\partial_t v(\vec{x},t)=-\nu\Delta v(\vec{x},t)+c(\vec{x},t)v(\vec{x},t),$$

where $c(\vec{x},t)=(1/2\nu)\Omega(\vec{x},t)$ follows from the previous formulas. In the above, $d\mu_{(\vec{x},t)}^{(\vec{y},s)}(\omega)$ is the conditional Wiener measure over sample paths of the standard Brownian motion.

Solutions of Eqs. (18), upon suitable normalization, give rise to the Markovian diffusion process with the factorized probability density $\rho(\vec{x},t)=u(\vec{x},t)v(\vec{x},t)$, which interpolates between the boundary density data $\rho(\vec{x},0)$ and $\rho(\vec{x},T)$, with the forward and backward drifts of the process defined as follows:

$$\vec{b}(\vec{x},t)=2\nu\frac{\vec{\nabla}v(\vec{x},t)}{v(\vec{x},t)}, \qquad (19)$$

$$\vec{b}_*(\vec{x},t)=-2\nu\frac{\vec{\nabla}u(\vec{x},t)}{u(\vec{x},t)},$$

in the prescribed time interval $[0,T]$. The transition probability density of this process reads:

$$p(\vec{y},s,\vec{x},t)=k(\vec{y},s,\vec{x},t)\frac{v(\vec{x},t)}{v(\vec{y},s)}. \qquad (20)$$

Here, neither k [Eq. (17)] nor p [Eq. (20)] needs to be the fundamental solutions of appropriate parabolic equations; see, e.g., Ref. [31], where an issue of differentiability is analyzed.

The corresponding [since $\rho(\vec{x},t)$ is given] transition probability density (10) of the backward process has the form

$$p_*(\vec{y},s,\vec{x},t)=k(\vec{y},s,\vec{x},t)\frac{u(\vec{y},s)}{u(\vec{x},t)}. \qquad (21)$$

Obviously [31,36], in the time interval $0\leqslant s<t\leqslant T$ there holds

$$u(\vec{x},t)=\int u_0(\vec{y})k(\vec{y},s,\vec{x},t)d^3y$$

and

$$v(\vec{y},s)=\int k(\vec{y},s,\vec{x},T)v_T(\vec{x})d^3x.$$

By defining $\Phi_*=\ln u$, we immediately recover the traditional form of the Hopf-Cole transformation for Burgers velocity fields: $\vec{b}_*=-2\nu\nabla\Phi_*$. In the special case of the standard free Brownian motion, there holds $\vec{b}(\vec{x},t)=\vec{0}$ while $\vec{b}_*(\vec{x},t)=-2\nu\vec{\nabla}\ln\rho(\vec{x},t)$.

Our discussion provides a complete identification of the stochastic diffusion process underlying *both* the deterministically forced Burgers velocity dynamics and the related matter transport (14), the latter in terms of suitable density fields. The generalization of the Hopf-Cole procedure to this case involves a powerful methodology of the Feynman-Kac kernel functions and yields exact formulas for solutions for the forced Burgers equation. Let us stress that the connection between the Burgers equation and the generalized (forward) heat equation is not merely a formal trick that generates solutions to the nonlinear problem. The forward equation (18), in fact, carries a complete information about the implicit *backward stochastic evolution*, that is, a Markov diffusion process for which the Burgers-velocity driven transport is appropriate. Notice that the transition probability density (21) obeys the familiar Chapman-Kolmogorov formula. If we wish to analyze a concrete density field governed by this process, any two boundary density data $\rho(\vec{x},0)$ and $\rho(\vec{x},T)$ allow one to deduce the ultimate form of the (more traditional, forward) diffusion process (20), by means of the Schrödinger boundary data problem [31,36]. Then, the adjoint pair of equations (18) gives all details of the dynamics, with (19)–(21) as a necessary consequence. On the other hand, the presented discussion implies a direct import of the shock-type matter density profiles to the general nonequilibrium statistical physics of diffusion-type processes.

II. PROBLEM OF NONCONSERVATIVE FORCING OF BURGERS VELOCITY FIELDS

By embedding the Burgers equation in the Schrödinger interpolation framework, we could consistently handle random transport that is governed by gradient velocity fields and gradient-type external conservative forces. The natural question at this point is how to incorporate the nongradient (rotational, for example) velocity fields and especially the nonconservative forces. This question may be addressed without reservations only in the context of the forced Burgers equation. Recall that the Hopf-Cole transformation is applicable only in the case of gradient velocity fields. Moreover, the involved Schrödinger interpolation framework extends the issue to the domain of nonequilibrium random phenomena, where standard Smoluchowski diffusions [32] are normally discussed in the case of conservative force fields (and drifts in consequence).

Remark: Strikingly, an investigation of typical nonconservative, e.g., electromagnetically, forced diffusions has not been much pursued in the literature, although an issue of deriving the Smoluchowski-Kramers equation (and possibly its large friction limit) from the Langevin-type equation for the charged Brownian particle in the general electromagnetic field has been relegated in Ref. [45], Chap. 6.1, to the status of the innocent-looking exercise. On the other hand, the diffusion of realistic charges in dilute ionic solutions creates a number of additional difficulties due to the apparent Hall mobility in terms of mean currents induced by the electric field (once assumed to act upon the system); see, e.g., [46–48]. In connection with the electromagnetic forcing of diffusing charges, the gradient field assumption imposes a severe limitation if we account for typical (nonzero circulation) features of the classical motion due to the Lorentz force, with or without the random perturbation component. The purely electric forcing is simpler to handle, since it has a definite gradient field realization; see, e.g., [49] for a recent discussion of related issues. The major obstacle with respect to our previous (Sec. I) discussion is that, if we wish to regard either the force $\vec{F}$ [Eqs. (8) and (12)] or drifts $\vec{b}$, $\vec{b}_*$ to have an electromagnetic origin, then necessarily we need to pass from conservative to nonconservative fields. This subject matter has not been significantly exploited so far in the nonequilibrium statistical physics literature.

With this additional (via the Burgers equation) motivation, let us analyze how the gradient velocity field (and conservative force field) assumption can be relaxed and nonetheless the exact solutions to the Burgers equation can be obtained, *both* in the unforced and forced cases, while involving the primordial Markovian diffusion process scenario.

It turns out that the crucial point of our previous discussion lies in a *proper* choice of the strictly positive and continuous (in an open space-time area) function $k(\vec{y},s,\vec{x},t)$, which, if we wish to construct a Markov process, has to satisfy the Chapman-Kolmogorov (semigroup composition) equation. It has led us to consider a pair of adjoint partial differential equations, (18), as an alternative to either (5) and (6) or (14) and (15).

The Feynman-Kac integration is predominantly utilized in the quantally oriented literature dealing with Schrödinger operators and their spectral properties [50,51]. We shall exploit some of results of this well developed theory. The pertinent Feynman-Kac potential $c(x,t)$ in Eqs. (17) and (18) is usually assumed to be a continuous and bounded-from-below function, but these restrictions can be substantially relaxed (unbounded functions are allowed in principle) if we wish to consider general Markovian diffusion processes and disregard an issue of the bound state spectrum and this of the ground state of the (self-adjoint) semigroup generator [25,30]. Actually, what we need is merely that the properties of $c(\vec{x},t)$ allow for the kernel k, (17), that is, positive and continuous. This property is crucial for the Schrödinger boundary-data problem analysis.

Taking for granted that suitable conditions are fulfilled [31,50], we can immediately associate with Eqs. (18) an integral kernel of the time-dependent semigroup [the exponential operator should be understood as a time-ordered expression, since in general $H(\tau)$ may not commute with $H(\tau')$ for $\tau\neq\tau'$]:

$$k(\vec{y},s,\vec{x},t)=\left[\exp\left(-\int_s^t H(\tau)d\tau\right)\right](\vec{y},\vec{x}), \tag{22}$$

where $H(\tau)=-\nu\Delta+c(\tau)$ is the pertinent semigroup generator. Then, by the Feynman-Kac formula [43], we get an expression (17) for the kernel, which in turn yields Eqs. (19)–(22); see, e.g., [31]. As mentioned before, Eq. (20) combined with Eq. (17) sets a probabilistic connection between the Wiener measure corresponding to the standard Brownian motion with $\vec{b}(\vec{x},t)=\vec{0}$ and that for the diffusion process with a nonvanishing drift $\vec{b}(\vec{x},t),\mathrm{curl}\vec{b}=\vec{0}$.

Our main purpose is to generalize Eq. (22), so that the positive and continuous (semigroup) kernel function can be associated with stochastic diffusion processes, whose drifts are no longer gradient fields. In particular, the forcing is to be nonconservative.

Since we have no particular hints towards Feynman-Kac-type analysis of rotational motions, it seems instructive to invoke the framework of the Onsager-Machlup approach towards an identification of most probable paths associated with the underlying diffusion process [52–54]. In this context, the nonconservative model system has been investigated in Ref. [55]. Namely, an effectively two-dimensional Brownian motion was analyzed, whose three-dimensional forward drift $\vec{b}(\vec{x}),b_3=0$ in view of $\partial_x b_1\neq\partial_y b_2$, has $\mathrm{curl}\vec{b}\neq 0$. Then, by the standard variational argument with respect to the Wiener-Onsager-Machlup action [53,55],

$$I\{L(\dot{\vec{x}},\vec{x},t);t_1,t_2\}=\frac{1}{2\nu}\int_{t_1}^{t_2}\left\{\frac{1}{2}[\dot{\vec{x}}-\vec{b}(\vec{x},t)]^2+\nu\vec{\nabla}\cdot\vec{b}(\vec{x},t)\right\}dt, \tag{23}$$

the most probable trajectory, about which major contributions from (weighted) Brownian paths are concentrated, was found to be a solution of the Euler-Lagrange equations, which are formally identical to the equations of motion

$$\ddot{\vec{q}}_{cl}=\vec{E}+\dot{\vec{q}}_{cl}\times\vec{B} \tag{24}$$

of a classical particle of unit mass and unit charge moving in an electric field $\vec{E}$ and the magnetic field $\vec{B}$. The electric field [to be compared with Eq. (9)] is given by

$$\vec{E}=-\vec{\nabla}\Phi, \tag{25}$$

$$\Phi=-\frac{1}{2}(\vec{b}^2+2\nu\vec{\nabla}\cdot\vec{b}),$$

while the magnetic field has the only nonvanishing component in the z direction of R^3:

$$\vec{B}=\mathrm{curl}\vec{b}=\{0,0,\partial_x b_2-\partial_y b_1\}. \tag{26}$$

Clearly, $\vec{B}=\mathrm{curl}\vec{A}$, where $\vec{A}\doteq\vec{b}$ is the electromagnetic vector potential. The simplest example is a notorious constant magnetic field defined by $b_1(\vec{x})=-(B/2)x_2,b_2(\vec{x})=(B/2)x_1$.

One immediately realizes that the Fokker-Planck equation in this case is incompatible with traditional intuitions underlying the Smoluchowski-drift identification: the forward drift is *not* proportional to an external force, but to an electromagnetic potential. Nevertheless, the variational information drawn from the Onsager-Machlup Lagrangian involves the Lorentz force-driven trajectory. Hence, some principal effects of the electromagnetic forcing are present in the diffusing system, whose drifts display an ''unphysical'' (gauge dependent) form.

On the other hand, if we accept this ''unphysical'' random motion to yield the representation with the nongradient drift $\vec{A}$: $d\vec{X}(t)=\vec{A}(\vec{X}(t),t)dt+\sqrt{2\nu}d\vec{W}(t)$, and consider the corresponding pair (5) and (6) of adjoint diffusion equations with $\vec{A}(\vec{x},t)$ replacing $\vec{b}(\vec{x},t)$, then Eq. (8) tells us that

$$(D_+^2\vec{X})(t)=\partial_t\vec{A}+(\vec{A}\cdot\vec{\nabla})\vec{A}+\nu\Delta\vec{A} = -\frac{B^2}{4}\{x_1,x_2,0\}=-\vec{E}(\vec{x}), \tag{27}$$

where $\vec{E}(\vec{x})=(B^2/4)\{x_1,x_2,0\}$, if calculated from Eqs. (25).

We thus arrive at the purely electric forcing with reversed sign [if compared with that coming from the Onsager-Machlup argument (25)] and, somewhat surprisingly, there is no impact of the previously discussed magnetic motion on the level of dynamical constraints [Eqs. (8) and (13)]. The adopted recipe is thus incapable of producing the magnetically forced diffusion process that conforms with arguments of Sec. I. Our toy model is inappropriate and a more sophisticated route must be adopted.

Below, we shall invoke the Feynman-Kac kernel idea (22) [31]. This approach has the clear advantage of elucidating the generic issues that hamper attempts to describe the diffussion processes governed by nonconservative (and electromagnetic in particular) force fields. The Burgers equation and the problem of its nongradient solutions will appear residually as a byproduct of the more general discussion.

Usually, the self-adjoint semigroup generators attract the attention of physicists in connection with the Feynman-Kac formula. Since electromagnetic fields provide the most conventional examples of nonconservative forces, we shall concentrate on their impact on random dynamics.

A typical route towards incorporating electromagnetism comes from quantal motivations via the minimal electromagnetic coupling recipe which preserves the self-adjointness of the generator (Hamiltonian of the system). As such, it constitutes a part of the general theory of Schrödinger operators. A rigorous study of operators of the form $-\triangle+V$ has become a well developed mathematical discipline [50]. The study of Schrödinger operators with magnetic fields, typically of the form $-(\nabla-i\vec{A})^2+V$, is less advanced, although specialized chapters on the magnetic field issue can be found in monographs devoted to functional integration methods [50,56], mostly in reference to seminal papers [57,58].

From the mathematical point of view, it is desirable to deal with magnetic fields that go to zero at infinity, which is certainly acceptable on physical grounds as well. The constant magnetic field (see, e.g., our previous considerations) does not meet this requirement, and its notorious usage in the literature makes us (at the moment) decline the asymptotic assumption and inevitably fall into a number of serious complications.

One obvious obstacle can be seen immediately by taking advantage of the existing results [57]. Namely, an explicit expression for the Feynman-Kac kernel in a constant magnetic field, introduced through the minimal electromagnetic coupling assumption $H(\vec{A})=-\frac{1}{2}(\vec{\nabla}-i\vec{A})^2$, is available (up to irrelevant dimensional constants):

$$\exp[-tH(\vec{A})](\vec{x},\vec{y})=\frac{B}{4\pi\,\sinh(\frac{1}{2}Bt)}\left(\frac{1}{2\pi t}\right)^{1/2} \times\exp\left\{-\frac{1}{2t}(x_3-y_3)^2-\frac{B}{4}\coth\left(\frac{B}{2}t\right) \times[(x_2-y_2)^2+(x_1-y_1)^2] -i\frac{B}{2}(x_1y_2-x_2y_1)\right\}. \tag{28}$$

Clearly, it is *not* real (hence *nonpositive* and directly at variance with the major demand in the Schrödinger interpolation problem, as outlined in Sec. I), except for directions $\vec{y}$ that are parallel to a chosen $\vec{x}$.

Consequently, a bulk of the well developed mathematical theory is of no use for our purposes and new techniques must be developed for a consistent description of the electromagnetically forced diffusion processes along the lines of Sec. I, i.e., within the framework of Schrödinger's interpolation problem. Also, another approach is necessary to generate solutions of the Burgers equation that are not in the gradient form.

III. FORCING VIA FEYNMAN-KAC SEMIGROUPS

The conditional Wiener measure $d\mu_{(\vec{x},t)}^{(\vec{y},s)}(\vec{\omega})$ appearing in the Feynman-Kac kernel definition (17), if unweighted [set $c(\vec{\omega}(\tau),\tau)=0$], gives rise to the familiar heat kernel. This, in turn, induces the Wiener measure P_W of the set of all sample paths, which originate from $\vec{y}$ at time s and terminate (can be located) in the Borel set $A\in R^3$ after time $t-s$: $P_W[A]=\int_A d^3x\int d\mu_{(\vec{x},t)}^{(\vec{y},s)}(\vec{\omega})=\int_A d\mu$, where, for simplicity of notation, the $(\vec{y},t-s)$ labels are omitted and $\mu_{(\vec{x},t)}^{(\vec{y},s)}$ stands for the heat kernel.

Having defined an Itô diffusion $\vec{X}(t)=\int_0^t\vec{b}(\vec{x},u)du+\sqrt{2\nu}\vec{W}(t)$, we are interested in the analogous path measure: $P_{\vec{X}}[A]=\int_A dx\int d\mu_{(\vec{x},t)}^{(\vec{y},s)}(\vec{\omega}_{\vec{X}})=\int_A d\mu(\vec{X})$.

Under suitable (stochastic [32]) integrability conditions imposed on the forward drift, we have granted the absolute continuity $P_X\ll P_W$ of measures, which implies the existence of a strictly positive Radon-Nikodym density. Its canonical Cameron-Martin-Girsanov form [32,50], reads:

$$\frac{d\mu(\vec{X})}{d\mu}(\vec{y},s,\vec{x},t)=\exp\frac{1}{2\nu}\left[\int_s^t\vec{b}(\vec{X}(u),u)d\vec{X}(u) -\frac{1}{2}\int_s^t[\vec{b}(\vec{X}(u),u)]^2du\right]. \tag{29}$$

If we assume that drifts are gradient fields, $\mathrm{curl}\vec{b}=0$, then the Ito formula allows one to reduce otherwise troublesome stochastic integration in the exponent of Eq. (29) [50,56] to ordinary Lebesgue integrals:

$$\frac{1}{2\nu}\int_s^t \vec{b}(\vec{X}(u),u)d\vec{X}(u)=\Phi(\vec{X}(t),t)-\Phi(\vec{X}(s),s) - \int_s^t du\left(\partial_t\Phi+\frac{1}{2}\vec{\nabla}\cdot\vec{b}\right)(\vec{X}(u),u). \tag{30}$$

After inserting Eq. (30) into Eq. (29) and next integrating with respect to the conditional Wiener measure, on account of Eq. (9) we arrive at the standard form of the Feynman-Kac kernel (17). Notice that Eq. (30) establishes a probabilistic basis for logarithmic transformations (19) of forward and backward drifts: $b=2\nu\vec{\nabla}\ln v=2\nu\vec{\nabla}\Phi$, $b_*=-2\nu\vec{\nabla}\ln u =-2\nu\vec{\nabla}\Phi_*$. The forward version is commonly used in connection with the transformation of the Fokker-Planck equation into the generalized heat equation, [32,59]. The backward version is the Hopf-Cole transformation, mentioned in Sec. I, used to map the Burgers equation into the very same generalized heat equation as in the previous case [2,42].

However, presently we are interested in nonconservative drift fields, $\mathrm{curl}\vec{b}\neq 0$, and in that case the stochastic integral in Eq. (29) is the major source of computational difficulties [35,50,56], for nontrivial vector potential field configurations. It explains the virtual absence of magnetically forced diffusion problems in the nonequilibrium statistical physics literature.

At this point, some steps of the analysis performed in Ref. [60] in the context of the ''Euclidean quantum mechanics'' (cf. also [37]) are extremely useful. Let us emphasize that the electromagnetic fields we utilize are always meant to be ordinary Maxwell fields with *no* Euclidean connotations (see, e.g., Chap. 9 of Ref. [56] for the Euclidean version of Maxwell theory).

Let us consider a gradient drift-field diffusion problem according to Sec. I, with Eqs. (17) and (30) involved and thus an adjoint pair (18) of parabolic equations completely defining the Markovian diffusion process. Furthermore, let $\vec{A}(\vec{x})$ be the time-independent vector potential for the Maxwellian magnetic field $\vec{B}=\mathrm{curl}\vec{A}$. We pass from the gradient realization of drifts to the new one, generalizing Eq. (19), for which the following decomposition into the gradient and nonconservative part is valid:

$$\vec{b}(\vec{x},t)=2\nu\vec{\nabla}\Phi(\vec{x},t)-\vec{A}(\vec{x}). \tag{31}$$

We denote $\theta(\vec{x},t)\doteq\exp[\Phi(\vec{x},t)]$ and admit that Eq. (31) is a forward drift of an Itô diffusion process with a stochastic differential $d\vec{X}(t)=[2\nu(\nabla\theta/\theta)-\vec{A}]dt+\sqrt{2\nu}d\vec{W}(t)$. On purely formal grounds, we deal here with an example of the Cameron-Martin-Girsanov transformation of the forward drift of a given Markovian diffusion process and we are entitled to ask for a corresponding measure transformation (29).

To this end, let us furthermore *assume* that $\theta(\vec{x},t)=\theta$ solves a partial differential equation

$$\partial_t\theta=-\nu\left(\nabla-\frac{1}{2\nu}\vec{A}(\vec{x})\right)^2\theta+c(\vec{x},t)\theta \tag{32}$$

with the notation $c(\vec{x},t)=(1/2\nu)\Omega(\vec{x},t)$ patterned after Eq. (9). Then, by using the Ito calculus and Eqs. (31) and (32) on the way (see, e.g., Ref. [60]), we can rewrite Eq. (29) as follows:

$$\frac{d\mu(\vec{X})}{d\mu}(\vec{y},s,\vec{x},t)=\exp\frac{1}{2\nu}\left[\int_s^t\left(2\nu\frac{\vec{\nabla}\theta}{\theta}-\vec{A}\right)(\vec{X}(u),u)d\vec{X}(u)-\frac{1}{2}\int_s^t\left(2\nu\frac{\vec{\nabla}\theta}{\theta}-\vec{A}\right)^2(\vec{X}(u),u)du\right]$$
$$=\frac{\theta(\vec{X}(t),t)}{\theta(\vec{X}(s),s)}\exp\left[-\frac{1}{2\nu}\int_s^t[\vec{A}(u)d\vec{X}(u)+\nu(\vec{\nabla}\cdot\vec{A})(\vec{X}(u))du+\Omega(\vec{X}(u),u)du]\right], \tag{33}$$

where $\vec{X}(s)=\vec{y},\vec{X}(t)=\vec{x}$.

More significant observation is that the Radon-Nikodym density (33), if integrated with respect to the conditional Wiener measure, gives rise to the Feynman-Kac kernel (22) of the *non-self-adjoint* semigroup (suitable integrability conditions need to be respected here as well [60]), with the generator $H_{\vec{A}}=-\nu[\nabla-(1/2\nu)\vec{A}(\vec{x})]^2+c(\vec{x},t)$ defined by the right-hand side of Eq. (32):

$$\partial_t\theta(\vec{x},t)=H_{\vec{A}}\theta(\vec{x},t)=\left[-\nu\Delta+\vec{A}(\vec{x})\cdot\vec{\nabla}+\frac{1}{2}(\vec{\nabla}\cdot\vec{A}(\vec{x}))-\frac{1}{4\nu}[\vec{A}(\vec{x})]^2+c(\vec{x},t)\right]\theta(\vec{x},t)$$
$$=-\nu\Delta\theta(\vec{x},t)+\vec{A}(\vec{x})\cdot\vec{\nabla}\theta(\vec{x},t)+c_{\vec{A}}(\vec{x},t)\theta(\vec{x},t). \tag{34}$$

Here

$$c_A(\vec{x},t)=c(\vec{x},t)+\frac{1}{2}(\nabla\vec{A})(\vec{x})-\frac{1}{4\nu}[\vec{A}(\vec{x})]^2. \qquad (35)$$

An adjoint parabolic partner of Eq. (34) reads

$$\partial_t\theta_*=-H_{\vec{A}}^*\theta_*=\nu\Delta\theta_*+\vec{\nabla}\cdot[\vec{A}(\vec{x})\theta_*]-c_A(\vec{x},t)\theta_*$$
$$=\nu\left[\vec{\nabla}+\frac{1}{2\nu}\vec{A}(\vec{x})\right]^2\theta_*-c(\vec{x},t)\theta_*. \qquad (36)$$

Consequently, our assumptions [Eqs. (31) and (32)] involve a generalization of the adjoint parabolic system (18) to a new adjoint one comprising Eqs. (32) and (36). Obviously, the original form of Eq. (18) is immediately restored by setting $\vec{A}=\vec{0}$, and executing obvious replacements $\theta_*\to u$, $\theta\to v$.

Let us emphasize again that, in contrast to Ref. [62], where the non-Hermitian generator $2\nu H_{\vec{A}}$, Eq. (32), has been introduced as "the Euclidean version of the Hamiltonian" $H=-2\nu^2[\nabla-(i/2\nu)\vec{A}]^2+\Omega$, our electromagnetic fields stand for solutions of the usual Maxwell equations and *are not* Euclidean at all.

As long as the coefficient functions (both additive and multiplicative) of the adjoint parabolic system (34) and (36) are not specified, we remain within a general theory of positive solutions for parabolic equations with unbounded coefficients (of particular importance, if we do not impose any asymptotic falloff restrictions) [30,61–63]. The fundamental solutions, if their existence can be granted, usually exist on space-time strips, and generally do not admit unbounded time intervals. We shall disregard these issues at the moment, and assume the existence of fundamental solutions without any reservations.

By exploiting the rules of functional (Malliavin, variational) calculus, under an assumption that we deal with a diffusion (in fact, Bernstein) process associated with an adjoint pair (34) and (35), it has been shown in Ref. [60] that *if* the forward conditonal derivatives of the process exist, then $(D_+\vec{X})(t)=2\nu(\nabla\theta/\theta)-\vec{A}=\vec{b}(\vec{x},t)$, Eq. (32), and

$$(D_+^2\vec{X})(t)=(D_+\vec{X})(t)\times\mathrm{curl}\vec{A}(\vec{x})+\nabla\Omega(\vec{x},t)$$
$$+\nu\ \mathrm{curl}[\mathrm{curl}\vec{A}(\vec{x})], \qquad (37)$$

where $\vec{X}(0)=0$, $\vec{X}(t)=\vec{x}$, $\times$ denotes the vector product in R^3, and $2\nu c=\Omega$.

Since $\vec{B}=\mathrm{curl}\vec{A}=\mu_0\vec{H}$, we identify in the above the standard Maxwell equation for $\mathrm{curl}\vec{H}$ comprising magnetic effects of electric currents in the system: $\mathrm{curl}\vec{B}=\mu_0[\dot{\vec{D}}+\sigma_0\vec{E}+\vec{J}_{ext}]$, where $\vec{D}=\epsilon_0\vec{E}$ while $\vec{J}_{ext}$ represents external electric currents. In case of $\vec{E}=\vec{0}$, the external currents only would be relevant. A demand curl $\mathrm{curl}\vec{A}=\vec{\nabla}(\vec{\nabla}\vec{A})-\Delta\vec{A}=0$ corresponds to a total absence of such currents, and the Coulomb gauge choice $\vec{\nabla}\cdot\vec{A}=0$ would leave us with harmonic functions $\vec{A}(\vec{x})$.

Consequently, a correct expression for the magnetically implemented Lorentz force has appeared on the right-hand side of the forward acceleration formula (37), with the forward drift (31) replacing the classical particle velocity $\dot{\vec{q}}$ of the classical formula (24).

The above discussion implicitly involves quite sophisticated mathematics; hence it is instructive to see that we can bypass the apparent complications by directly invoking the universal definitions (7) and (11) of conditional expectation values, which are based on exploitation of the Itô formula only. Obviously, we assume that the Markovian diffusion process with well defined transition probability densities $p(\vec{y},s,\vec{x},t)$ and $p_*(\vec{y},s,\vec{x},t)$, does exist.

We shall utilize an obvious generalization of canonical definitions (19) of both forward and backward drifts of the diffusion process defined by the adjoint parabolic pair (18), as suggested by Eq. (31) with $\vec{A}=\vec{A}(\vec{x})$:

$$\vec{b}=2\nu\frac{\vec{\nabla}\theta}{\theta}-\vec{A},\quad \vec{b}_*=-2\nu\frac{\vec{\nabla}\theta_*}{\theta_*}-\vec{A}. \qquad (38)$$

We also demand that the corresponding adjoint equations (34) and (36) *are* solved by θ and θ_*, respectively.

Taking for granted that identities $(D_+\vec{X})(t)=\vec{b}(\vec{x},t),\vec{X}(t)=\vec{x}$, and $(D_-\vec{X})(t)=\vec{b}_*(\vec{x},t)$ hold true, we can easily evaluate the forward and backward accelerations [substitute Eq. (38), and exploit Eqs. (34) and (36)]:

$$(D_+\vec{b})(\vec{X}(t),t)=\partial_t\vec{b}+(\vec{b}\cdot\vec{\nabla})\vec{b}+\nu\Delta\vec{b}$$
$$=\vec{b}\times\vec{B}+\nu\ \mathrm{curl}\vec{B}+\vec{\nabla}\Omega \qquad (39)$$

and

$$(D_-\vec{b}_*)(\vec{X}(t),t)=\partial_t\vec{b}_*+(\vec{b}_*\cdot\vec{\nabla})\vec{b}_*-\nu\Delta\vec{b}_*$$
$$=\vec{b}_*\times\vec{B}-\nu\mathrm{curl}\vec{B}+\vec{\nabla}\Omega. \qquad (40)$$

Let us notice that the forward and backward acceleration formulas *do not* coincide as was the case before [cf. Eqs. (8) and (12)]. There is a definite time asymmetry in the local description of the diffusion process in the presence of general magnetic fields, unless $\mathrm{curl}\vec{B}=0$. The quantity which is explicitly time-reversal invariant can be easily introduced:

$$\vec{v}(\vec{x},t)=\tfrac{1}{2}(\vec{b}+\vec{b}_*)(\vec{x},t) \qquad (41)$$

$$\Rightarrow\tfrac{1}{2}(D_+^2+D_-^2)(\vec{X}(t))=\vec{v}\times\vec{B}+\vec{\nabla}\Omega.$$

As yet there is no trace of Lorentzian electric forces, unless extracted from the term $\vec{\nabla}\Omega(\vec{x},t)$. We shall accomplish this step in Sec. IV.

For a probability density $\theta_*\theta=\rho$ of the related Markovian diffusion process [31,36], we would have fulfilled both the Fokker-Planck and the continuity equations: $\partial_t\rho=\nu\Delta\rho-\vec{\nabla}(\vec{b}\rho)=-\vec{\nabla}(\vec{v}\rho)=-\nu\Delta\rho-\vec{\nabla}(\vec{b}_*\rho)$, as before (cf. Sec. I).

In the above, Eq. (40) can be regarded as the Burgers equation with a general external magnetic (plus other external force contributions if necessary) forcing, and its definition is an outcome of the underlying mathematical structure related to the adjoint pair (32) and (36) of parabolic equations.

Our construction shows that solutions of the magnetically forced Burgers equation (40) are given in the form (38). In reverse, the mere assumption about the decomposition of drifts (38) into the gradient and nongradient part implies that the corresponding evolution equation (40) is the Burgers equation with the nonconservative forcing. The force term has a specific Lorentz form. Although we invoke electromagnetism, the decomposition (38) can be regarded to refer to an abstract nongradient component. In analogy to the previous Onsager-Machlup example, Eqs. (24)–(28), the fictituous Lorentz force term would arise anyway.

IV. SCHRÖDINGER'S INTERPOLATION IN A CONSTANT MAGNETIC FIELD AND QUANTALLY INSPIRED GENERALIZATIONS

Presently, we shall confine our attention to the simplest case of a constant magnetic field, defined by the vector potential $\vec{A}=\{-(B/2)x_2,+(B/2)x_1,0\}$. Here, $\vec{B}=\{0,0,B\}$, $\vec{\nabla}\cdot\vec{A}=0$, and $\mathrm{curl}\vec{B}=\vec{0}$, which significantly simplifies formulas (31)–(41).

As emphasized before, most of our discussion was based on the existence assumption for fundamental solutions of the (adjoint) parabolic equations (32) and (36). For magnetic fields, which do not vanish at spatial infinities (hence for our ''simplest'' choice), the situation becomes rather complicated. Namely, an expression for

$$c_{\vec{A}}(\vec{x},t)=c(\vec{x},t)-\frac{B^2}{16\nu}(x_1^2+x_2^2) \tag{42}$$

includes a *repulsive* harmonic oscillator contribution.

For the existence of a well defined Markovian diffusion process it appears necessary that a nonvanishing contribution from an unbounded from above $c(\vec{x},t)$ would counterbalance the harmonic repulsion. To see that this *must be* the case, let us formally constrain $\theta(\vec{x},t)=\exp[\Phi(\vec{x},t)]$ to yield [in accordance with Eq. (9)] the identity:

$$c(\vec{x},t)=\partial_t\Phi+\nu[\vec{\nabla}\Phi]^2+\nu\triangle\Phi=0. \tag{43}$$

Then, we deal with the simplest version of the adjoint system (34) and (36) where, in view of $\vec{\nabla}\cdot\vec{A}=0=c$, there holds:

$$\partial_t\theta=-\nu\left[\vec{\nabla}-\frac{1}{2\nu}\vec{A}\right]^2\theta=-\nu\Delta\theta+\vec{A}\cdot\vec{\nabla}\theta-\frac{1}{4\nu}[\vec{A}]^2\theta, \tag{44}$$

$$\partial_t\theta_*=\nu\left[\vec{\nabla}+\frac{1}{2\nu}\vec{A}\right]^2\theta_*=\nu\Delta\theta_*+\vec{A}\cdot\vec{\nabla}\theta_*+\frac{1}{4\nu}[\vec{A}]^2\theta_*.$$

With our choice, $\mathrm{curl}\vec{A}=\{0,0,B\}$, Eqs. (44) *do not* possess a fundamental solution, which would be well defined for *all* $(\vec{x},t)\in R^3\times R^+$: everything because of the harmonic repulsion term in the forward parabolic equation. We can prove (this purely mathematical argument is not reproduced in the present paper) that the function

$$k(\vec{y},s,\vec{x},t)=\frac{B}{4\pi\,\sin[\frac{1}{2}B(t-s)]}\left(\frac{1}{2\pi(t-s)}\right)^{1/2}\times\exp\left\{-\frac{1}{2(t-s)}(x_3-y_3)^2-\frac{B}{4}\cot\left(\frac{B}{2}(t-s)\right)[(x_2-y_2)^2+(x_1-y_1)^2]-\frac{B}{2}(x_1y_2-x_2y_1)\right\} \tag{45}$$

only when restricted to times $t-s\leqslant\pi/B$ is an acceptable example of a *unique* positive (actually, positivity extends to times $t-s\leqslant 2\pi/B$) fundamental solution of the system (43), (rescaled to yield $\nu\rightarrow\frac{1}{2}$). Here, formally, Eq. (45) can be obtained from the expression (28) by the replacement $\vec{A}\rightarrow -i\vec{A}$.

An immediate insight into a harmonic repulsion obstacle can be achieved after an x-y plane rotation of Cartesian coordinates: $x_1'=x_1\cos(\omega t)-x_2\sin(\omega t),x_2'=x_1\sin(\omega t)+x_2\cos(\omega t),x_3'=x_3,t'=t$, with $\omega=B/4\sqrt{\nu}$. Then, Eqs. (44) get transformed into an adjoint pair:

$$\partial_{t'}\theta=-\nu\Delta'\theta-\omega^2(x_1'^2+x_2'^2)\theta, \tag{46}$$

$$\partial_{t'}\theta_*=\nu\Delta'\theta_*+\omega^2(x_1'^2+x_2'^2)\theta_*.$$

Notice that the transformation $\omega\rightarrow i\omega$ would replace repulsion in Eqs. (46) by harmonic attraction. On the other hand, we can get rid of the repulsive term by assuming that $c(\vec{x},t)$ [Eq. (42)] does not identically vanish. For example, we can formally demand that, instead of Eq. (43), $c(\vec{x},t)=+(B^2/8\nu)(x_1^2+x_2^2)$ plays the role of an electric potential. Then, harmonic attraction replaces repulsion in the final form of Eqs. (34) and (36).

As a byproduct, we are given a transition probability density of the diffusion process governed by the adjoint system [cf. Eq. (27)]:

$$\partial_t\theta=-\nu\Delta\theta+\vec{A}\cdot\vec{\nabla}\theta, \tag{47}$$

$$\partial_t\theta_*=\nu\Delta\theta_*+\vec{A}\cdot\vec{\nabla}\theta_*.$$

with $\vec{A}=(B/2)\{-x_2,x_1,0\}$. Namely, by means of the previous x-y plane rotation, Eqs. (47) are transformed into a pair of time adjoint heat equations:

$$\partial_{t'}\theta=-\nu\Delta'\theta, \quad \partial_{t'}\theta_*=\nu\triangle'\theta_*, \tag{48}$$

whose fundamental solution is the standard heat kernel.

Finding explicit analytic solutions of rather involved equations (34) and (36) is a formidable task on its own, in contrast to much simpler unforced or conservatively forced dynamics issue.

Interestingly, we can produce a number of examples by invoking the quantum Schrödinger dynamics. This quantum inspiration has been proved to be very useful in the past [36,37]. At this point, we shall follow the idea of Ref. [31], where the strategy developed for solving the Schrödinger boundary data problem has been applied to quantally induced stochastic processes (e.g., Nelson's diffusions [35,38]). They were considered as a particular case of the general theory appropriate for nonequilibrium statistical physics processes as governed by the adjoint pair (18), and exclusively in conjunction with Born's statistical postulate in quantum theory.

The Schrödinger picture quantum evolution is then consistently representable as a Markovian diffusion process. All that follows from the previously outlined Feynman-Kac kernel route [31,32,35,36,38,40,41], based on exploiting the adjoint pairs of parabolic equations. However, the respective semigroup theory has been developed for pure gradient drift fields, hence without reference to any impact of electromagnetism on the pertinent diffusion process, and electromagnetism is definitely ubiquitous in the world of quantum phenomena.

Let us start from an ordinary Schrödinger equation for a charged particle in an arbitrary external electromagnetic field, in its standard dimensional form. To conform with the previous notation let us absorb the charge e and mass m parameters in the definition of $\vec{A}(\vec{x})$ and the potential $\phi(\vec{x})$, e.g., we consider B instead of $(e/m)B$ and ϕ instead ϕ/m. Additionally, we set ν instead of $(\hbar/2m)$. Then, we have

$$i\partial_t\psi(\vec{x},t)=-\nu\left(\vec{\nabla}-\frac{i}{2\nu}\vec{A}\right)^2\psi(\vec{x},t)+\frac{1}{2\nu}\phi(\vec{x})\psi(\vec{x},t). \tag{49}$$

The standard Madelung substitution $\psi=\exp(R+iS)$ allows one to introduce the real functions $\theta=\exp(R+S)$ and $\theta_*=\exp(R-S)$ instead of complex ones $\psi,\bar{\psi}$. They are solutions of an adjoint parabolic system (34) and (36), where the impact of Eq. (49) is encoded in a specific functional form of the otherwise arbitrary potential $c(\vec{x},t)$:

$$c(\vec{x},t)=\frac{1}{2\nu}\Omega(\vec{x},t)=\frac{1}{2\nu}[2Q(\vec{x},t)-\phi(\vec{x})], \tag{50}$$

$$Q(\vec{x},t)=2\nu^2\frac{\Delta\rho^{1/2}(\vec{x},t)}{\rho^{1/2}(\vec{x},t)}=2\nu^2\{\Delta R(\vec{x},t)+[\vec{\nabla}R(\vec{x},t)]^2\}.$$

The quantum probability density $\rho(\vec{x},t)=\psi(\vec{x},t)\overline{\psi}(\vec{x},t)=\theta(\vec{x},t)\theta_*(\vec{x},t)$ displays a factorization $\rho=\theta\theta_*$ in terms of solutions of adjoint parabolic equations, which we recognize to be characteristic for probabilistic solutions (Markov diffusion processes) of the Schrödinger boundary data problem (cf. Sec. I) [31,32,36,40]. It is easy to verify the validity of the Fokker-Planck equation whose forward drift has the form (38). Also, Eqs. (39) and (40) do follow with $\Omega=2Q-\phi$.

By defining $\vec{E}=-\vec{\nabla}\phi$ [with ϕ utilized instead of $(e/m)\phi$], we immediately arrive at the complete Lorentz force contribution in all acceleration formulas (before, we have used $\mathrm{curl}\vec{B}=0$):

$$\partial_t\vec{b}+(\vec{b}\cdot\vec{\nabla})\vec{b}+\nu\Delta\vec{b}=\vec{b}\times\vec{B}+\vec{E}+\nu\,\mathrm{curl}\vec{B}+2\vec{\nabla}Q, \tag{51}$$

$$\partial_t\vec{b}_*+(\vec{b}_*\cdot\vec{\nabla})\vec{b}_*-\nu\Delta\vec{b}_*=\vec{b}_*\times\vec{B}+\vec{E}-\nu\,\mathrm{curl}\vec{B}+2\vec{\nabla}Q.$$

Moreover, the velocity field named the current velocity of the flow, $\vec{v}=\frac{1}{2}(\vec{b}+\vec{b}_*)$, enters the familiar local conservation laws (see also [32] for a discussion of how the "quantum potential" Q affects such laws in case of the standard Brownian motion and Smoluchowski-type diffusion processes)

$$\partial\rho=-\vec{\nabla}(\vec{v}\rho), \tag{52}$$

$$\partial_t\vec{v}+(\vec{v}\cdot\vec{\nabla})\vec{v}=\vec{v}\times\vec{B}+\vec{E}+\vec{\nabla}Q.$$

A comparison with Eqs. (33)–(43) shows that Eqs. (50)–(53) can be regarded as the specialized version of the general external forcing problem with an explicit electromagnetic (Lorentz force-inducing) contribution and an arbitrary term of nonelectromagnetic origin, which we denote by $c(\vec{x},t)$ again. Obviously, c is represented in Eq. (50), by $(1/\nu)Q(\vec{x},t)$.

We have therefore arrived at the following ultimate generalization of the adjoint parabolic system (18), that encompasses the nonequilibrium statistical physics and essentially quantum evolutions on an equal footing (with no clear-cut discrimination between these options, as in Ref. [31]) and gives rise to an external (Lorentz) electromagnetic forcing:

$$\partial_t\theta(\vec{x},t)=\left[-\nu\left(\vec{\nabla}-\frac{1}{2\nu}\vec{A}\right)^2-\frac{1}{2\nu}\phi(\vec{x})+c(\vec{x},t)\right]\theta(\vec{x},t), \tag{53}$$

$$\partial_t\theta_*(\vec{x},t)=\left[\nu\left(\vec{\nabla}+\frac{1}{2\nu}\vec{A}\right)^2+\frac{1}{2\nu}\phi(\vec{x})-c(\vec{x},t)\right]\theta_*(\vec{x},t).$$

A subsequent generalization encompassing time-dependent electromagnetic fields is immediate.

The adjoint parabolic pair of equations (53) can thus be regarded to determine a Markovian diffusion process in exactly the same way as Eq. (18) did. If only a suitable choice of vector and scalar potentials in Eqs. (53) guarrantees a continuity and positivity of the involved semigroup kernel [take the Radon-Nikodym density of the form (33), with $\Omega\rightarrow-\phi+\Omega$, and integrate with respect to the conditional Wiener measure], then the mere knowledge of such integral kernel suffices for the implementation of steps (18)–(22), with $u\rightarrow\theta_*$, $v\rightarrow\theta$. To this end it is not at all necessary that

$k(\vec{x},s,\vec{y},t)$ be a fundamental solution of Eqs. (53). A sufficient condition is that the semigroup kernel is a continuous (and positive) function. The kernel may not even be differentiable; see, e.g., Ref. [31] for a discussion of that issue which is typical for quantal situations.

After adopting Eqs. (53) as the principal dynamical ingredient of the electromagnetically forced Schrödinger interpolation, we must slightly adjust the emerging acceleration formulas. Namely, they have the form (51), but we need to replace $2Q(\vec{x},t)$ by, from now an arbitrary, potential $\Omega(\vec{x},t)=2\nu c(\vec{x},t)$. The second equation in Eqs. (53) also takes a new form:

$$\partial_t\vec{v}+(\vec{v}\cdot\vec{\nabla})\vec{v}=\vec{v}\times\vec{B}+\vec{E}+\vec{\nabla}(\Omega-Q); \qquad (54)$$

see, e.g., Ref. [32] for more detailed explanation of this step. The presence in Eqs. (53) of the density-dependent $-\vec{\nabla}Q$ term finds its origin in the identity $\vec{b}-\vec{b}_*=2\nu\nabla\rho(\vec{x},t)$ and is a necessary consequence of the involved (forced in the present case) Brownian motion; see, e.g., [39,64,65].

Finally, the second of equations (51) with Ω replacing $2Q$ is the most general form of the Burgers equation with an external forcing, where the electromagnetic (Lorentz force) contribution has been extracted for convenience. Solutions of this equation must be sought for in the form (38), which generalizes the logarithmic Hopf-Cole transformation to nongradient drift fields. Equations (53) are the associated parabolic partial differential (generalized heat) equations, which completely determine probabilistic solutions (Markovian diffusion processes) of the Schrödinger boundary data (interpolation) problem. In turn, for this particular random transport, the forced Burgers velocity fields play the role of backward drifts of the process.

V. OUTLOOK

Our discussion, albeit motivated by the issue of diffusive matter transport that is consistently driven by Burgers velocity fields (this extends both to the compressible and incompressible cases), has little to do with classical fluids. The emergence of shock pressure fronts is more natural in the compressible situation. This shock profile possibility (inherent to the Burgers equation) has been imported to the nonequilibrium statistical physics of random phenomena by exploring the idea of Schrödinger's interpolation problem and revealing its connection with the Burgers dynamics. That has been the subject of Sec. I.

The next important result (a preliminary discussion of rotational Burgers fields can be found in Ref. [23]) amounts to relaxing the gradient-field assumption (that is crucial for the validity of the Hopf-Cole transformation). In Secs. II and III we have analyzed the ways to generalize the Feynman-Kac kernel strategy so that the involved (drifts) velocity fields admit the nongradient form. Our analysis was perfomed with rather explicit electromagnetic connotations. Equations (34) and (36) generalize the adjoint pair (18) to diffusion processes with nongradient drifts (38).

As follows from Eq. (40), the very presence of the nongradient term in the decomposition (38) implies that the corresponding evolution equation for the velocity field (backward drift of the process) is the Burgers equation with the specific Lorentz-type forcing.

Section IV extends the discussion to quantally implemented diffusion processes, where the minimal electromagnetic coupling is a celebrated recipe. This quantal motivation allows to arrive at the adjoint system (53), that incorporates an electric contribution and allows one to define and solve the Burgers equation with the combined conservative and nonconservative (electromagnetic, in particular) forcing. Let us emphasize again that a transformation of the Burgers equation (whatever the force term is) into a generalized diffusion equation is not merely a formal linearization trick. This [1] ''nonlinear diffusion equation'' does indeed refer to a well defined stochastic diffusion process, but a complete information about its features is encoded in the involved parabolic equations.

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